The hamiltonian formulation of QCD in terms of angle variables

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Abstract

For the sake of eliminating gauge variant degrees of freedom we discuss the way to introduce angular variables in the hamiltonian formulation of QCD. On the basis of an analysis of Gauss' law constraints a particular choice is made for the variable transformation from gauge fields to angular field variables. The resulting formulation is analogous to the one of Bars in terms of corner variables. Therefore the corner or angle formulation may constitute an useful starting point for the investigation of the low energy properties of QCD in terms of gauge invariant degrees of freedom.

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One of the long standing problems in contemporary physics is understanding confinement of quarks and gluons from first principles. The difficulty in dealing with the infrared properties of QCD is on the one hand due to the non–linear gluonic interaction and on the other due to the constraints on the dynamics of the fundamental degrees of freedom which originate from the requirement of gauge invariance. In spite of the general belief that the non–linear interaction gives rise to confinement it has been conjectured recently that in fact the non–abelian constraints may be most important [1]. Aiming at an understanding of the low energy properties of QCD we should therefore try to develop approximations to the full QCD dynamics after the gauge variant degrees of freedom have been identified and isolated.

The aforementioned constraints are specified in terms of Gauss' law operators, which generate a compact group in each point in space, telling us that the gauge variant degrees of freedom are "angle" variables. In spite of this observation the choice of unphysical variables is to a large extent arbitrary due to the fact that field theory deals with an infinite number of degrees of freedom. Therefore various decompositions into unphysical "angle" variables and remaining physical variables are possible to arrive at the desired separation of unphysical degrees of freedom [1, 2]. Although successful in that respect the variables chosen in this way to parametrize the physical Hilbert space may be inadequate to account in a simple way for the dynamics relevant for the low energy properties of QCD. For further variable changes, on the other hand, the complexity of the so derived hamiltonians constitutes a basic obstacle.

In order to avoid this problem we start from the assumption that not only the unphysical but all variables are "angle" variables. The hamiltonian should therefore be expressed first in terms of these angular degrees of freedom before making a separation into gauge variant and gauge invariant ones. To find a suitable definition of "angle" variables in terms of gauge or electric fields we concentrate on an analysis of Gauss' law operators. It will be shown that the form of these operators suggests the introduction of "angles" which are non–locally related to the gauge fields. By a variable transformation the originally quantized gauge fields and electric fields in the hamiltonian can be replaced by "angle" and angular momentum operators respectively. The resulting formulation is analogous to the one in terms of corner variables obtained by Bars [3]. In contrast to similar approaches [1, 2] the separation into gauge variant and gauge invariant degrees of freedom is not made from the outset in the "angle" or corner variable formulation. Therefore it may constitute an useful starting point for the search of approximations to the full QCD dynamics intended to understand its nonperturbative aspects.

We consider a hamiltonian formulation of SU(N) gauge theories on a d–dimensional torus. Choosing the Weyl gauge $A_0 = 0$ we have the following hamiltonian density

$$\mathcal{H} = \sum_i \bar{\psi}(x)\gamma_i \left(i\partial_i + gA_i\right)\psi(x) + m\bar{\psi}(x)\psi(x) + \frac{1}{2}\sum_i E_i^a(x)E_i^a(x) + \frac{1}{2}\sum_{ij}\mathrm{Tr}\left\{F_{ij}F^{ij}\right\} \ .$$

Imposing periodic boundary conditions for the gauge and anti–periodic ones for the

fermion fields we quantize canonically ($E_i^a(x) = \partial_0 A_i^a(x)$)

$$
\begin{aligned}
\left[E_i^a(x), A_j^b(y)\right] &= -i\delta_{a,b}\delta_{ij}\delta^d(x-y); \quad a,b = 1,\dots,N^2-1;\ i,j = 1,\dots,d \\
\left\{\psi_{k,\alpha}^\dagger(x), \psi_{l,\beta}(y)\right\} &= \delta_{k,l}\delta_{\alpha,\beta}\delta^d(x-y); \quad k,l = 1,\dots,N;\ \alpha,\beta = \text{Spinor indices}
\end{aligned}
$$

where it is understood that the δ–functions are periodic, as well. Since we have not fixed the gauge classically, Gauss' law operator is the quantum mechanical generator of the gauge symmetry. It commutes with the hamiltonian and therefore physical eigenstates must be invariant under infinitesimal gauge transformations, which implies that they must be annihilated by the generators of the symmetry

$$
\begin{aligned}
G^a(x)|\,phys. > &= 0 & (1)\\
G^a(x) &= \sum_i \left[\partial_i E_i^a(x) + g f^{abc} A_i^b(x) E_i^c(x)\right] + g\psi^\dagger(x)\frac{\lambda^a}{2}\psi(x) & (2)\\
\left[G^a(x), G^b(y)\right] &= i g f^{abc} G^c(x)\delta^d(x-y)\ . & (3)
\end{aligned}
$$

Since the generators obey the Lie algebra of the gauge group it is understood that out of $d\cdot(N^2-1)$ gauge degrees of freedom only a set of N^2-1 "angle" variables in each point in space is changed by gauge transformations . Consequently the constraints are satisfied if these "angles" have been identified and Gauss' law operator has been transformed such that it is the angular momentum operator only with respect to these unphysical "angles". Physical states then correspond to s–wave states which are annihilated by these angular momentum operators and the hamiltonian after transformation will not contain the unphysical variables anymore.

Since we want to replace gauge and electric fields by "angles" and angular momenta in such a way that the constraints can easily be implemented, we study the form of Gauss' law operators in detail. In 1+1 dimensions the contribution in eq.(2)

$$
f^{abc} A^b(x) E^c(x) \tag{4}
$$

acts locally as an angular momentum operator on $N(N-1)$ "angle variables in either the gauge field or the elctric field representation. The missing $(N-1)$ "angle" variables could not be identified if this was the complete Gauss' law operator already. Therefore we must conclude that in 1+1 dimension the full number of (N^2-1) variables in each point in space can only be eliminated due to the presence of $\partial_x E(x)$ in the Gauss' law operators. This term not only distinguishes the gauge fields as source of the additional unphysical variables but also introduces a non-locality into the Gauss' law operators. Therefore it seems natural to assume that the "angle" variables which are unphysical are nonlocally related to the gauge field variables. Although this argument is rigorous only in 1+1 dimensions we assume it to be an useful hypothesis for introducing "angle" variables in any dimensions.

An expression for the gauge fields satisfying this requirement is[1]

$$
\begin{aligned}
A_i(x) &= \frac{i}{g} V_i(x) \partial_i V_i^\dagger(x) \quad \text{(no summation)} && (5)\\
V_i(x) &= P \exp\left[ig \int_0^{x_i} dz_i A_i(x_i^\perp, z_i) \right] && (6)\\
V_i(x) &= \exp\left[i\xi_i(x) \right], \quad \xi_i(x) = \xi_i^a(x) \frac{\lambda^a}{2}; \quad 0 < x_i \le L && (7)
\end{aligned}
$$

where $V_i(x)$ is a SU(N) matrix parametrized in terms of "angles" $\xi_i^a(x)$, P denotes path ordering and $x_i^\perp$ stands for all coordinates orthogonal to x_i. Since this definition together with the specific choice of paths in eq.(6) leads to an unique relation[2] between $\xi_i(x)$ and $A_i(x)$ a change of variables from $A_i(x), E_i(x)$ to $\xi_i(x)$ and the corresponding angular momenta $J_i(x)$ becomes feasible. Using eq.(5) we rewrite fermionic and magnetic part of the hamiltonian

$$
\bar{\psi}(x) \left\{ \gamma_i \left[i\partial_i + gA_i(x) \right] + m \right\} \psi(x) = \left[\bar{\psi}(x) U_i(x) \right] \left\{ \gamma_i i\partial_i + m \right\} \left[U_i^\dagger(x) \psi(x) \right] \;, \tag{8}
$$

$$
\begin{aligned}
[D_i, D_j] &= [i\partial_i + gA_i, i\partial_j + gA_j] = -U_i \left\{ \partial_i \left[\left(U_i^\dagger U_j \right) \partial_j \left(U_j^\dagger U_i \right) \right] \right\} U_i^\dagger \;, && (9)\\
\Rightarrow \mathrm{Tr}\left\{ F_{ij} F_{ij} \right\} &= \frac{-1}{g^2} \mathrm{Tr} \left\{ \partial_i \left[\left[\left(U_i^\dagger U_j \right) \partial_j \left(U_j^\dagger U_i \right) \right] \right] \left[\partial_i \left[\left(U_i^\dagger U_j \right) \partial_j \left(U_j^\dagger U_i \right) \right] \right]^\dagger \right\}.
\end{aligned}
$$

In order to reformulate the electric part of the hamiltonian which contains the conjugate momenta of the gauge fields, we introduce the angular momentum operators $J_k^c(z)$, the definition of which may be found in the appendix eq.(20). These operators generate translations in the space of "angles" ξ_k as may be seen from the commutation relations

$$
\begin{aligned}
\left[J_i^a(x), V_j(y) \right] &= \delta_{i,j} \delta^d(x-y) V_j(y) \frac{\lambda^a}{2} && (10)\\
\left[J_i^a(x), J_j^b(y) \right] &= \delta_{i,j} i f^{abc} J_i^c(x) \delta^d(x-y) \;.
\end{aligned}
$$

We note that due to the periodicity of V_i, J_i the δ–functions in these expressions are periodic, as well. Introducing furthermore the orthogonal matrices N_i

$$
\begin{aligned}
N_i^{ac}(x) &= \mathrm{Tr}\left\{ V_i^\dagger(x) \frac{\lambda^a}{2} V_i(x) \lambda^c \right\} && (11)\\
\left[J_i^b(z), N_j^{ac}(x) \right] &= i f^{bce} N_j^{ae}(x) \delta_{i,j} \delta^d(x-z)
\end{aligned}
$$

[1] Note that throughout the paper spatial indices are not summed over unless explicitly indicated.

[2] With the choice $0 < x_i \le L$ the "angles" are uniquely determined from gauge fields by eq.(6) if the derivative is taken from one side only $\partial_i f(x_i) = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left[f(x_i) - f(x_i - \epsilon) \right]; \quad 0 < x_i \le L$. In this way it is possible to work with periodic, although not continuous "angles" and angular momenta.

and using the identity eq.(21) stated in the appendix, we find for the electric part of the hamiltonian density the expression

$$\begin{aligned} E_i^a(x) &= g\int d^d z \delta^{d-1}(z_i^\perp - x_i^\perp)\theta(z_i - x_i)\theta(x_i)N_i^{ac}(x)J_i^c(z) \\ \frac{1}{2}E_i^a(x)E_i^a(x) &= \frac{g^2}{2}\int_{x_i}^L dz_i J_i^b(x_i^\perp, z_i)\int_{x_i}^L dz_i' J_i^b(x_i^\perp, z_i') \ . \end{aligned} \tag{12}$$

Collecting all the results the hamiltonian density reads

$$\begin{aligned} \mathcal{H} &= \sum_i \left[\bar\psi(x)U_i(x)\right]\left[\gamma_i i\partial_i + m\right]\left[U_i^\dagger(x)\psi(x)\right] \\ &+ \frac{g^2}{2}\sum_i\int_{x_i}^L dz_i J_i^b(x_i^\perp, z_i)\int_{x_i}^L dz_i' J_i^b(x_i^\perp, z_i') \\ &+ \frac{1}{2g^2}\sum_{ij}\mathrm{Tr}\left\{\left[\partial_i\left[\left(U_i^\dagger U_j\right)\partial_j\left(U_j^\dagger U_i\right)\right]\right]\left[\partial_i\left[\left(U_i^\dagger U_j\right)\partial_j\left(U_j^\dagger U_i\right)\right]\right]^\dagger\right\} \end{aligned} \tag{13}$$

which is the "angle" representation we have been looking for. Note that the locality of the hamiltonian has been lost although we have not been fixing the gauge yet. The electric part of the hamiltonian is non–local and it shows already the linear "potential" $|z_i - z_i'|$ characteristic for both the axial gauge formulation and the strong coupling limit in lattice gauge theory. We observe also that the hamiltonian in QED, corresponding to eq.(13), is obtained by dropping the summations over color indices which shows the similarity of abelian and non–abelian gauge theories in the "angle" formulation.
Finally we want to consider the form of Gauss' law operator in these variables. We find

$$g f^{abc}A_i^b(x)E_i^c(x) = -g\left[\partial_i N_i^{ad}(x)\right]\int dz_i\theta(z_i - x_i)\theta(x_i)J_i^d(x_i^\perp, z_i) \tag{14}$$

$$\begin{aligned} \partial_i E_i^a(x) &= g\left[\partial_i N_i^{ad}(x)\right]\int dz_i\theta(z_i - x_i)\theta(x_i)J_i^d(x_i^\perp, z_i) \\ &\quad + gN_i^{ad}(x)\int dz_i J_i^d(x_i^\perp, z_i)\partial_i\left[\theta(z_i - x_i)\theta(x_i)\right] \end{aligned} \tag{15}$$

and taking the sum of these two contributions and the charge density operator we obtain for the Gauss' law operators the following expression

$$G^a(x) = -g\sum_i\left[N_i^{ad}(x)J_i^d(x) - \delta(x_i)\int dz_i J_i^a(x_i^\perp, z_i)\right] + g\psi^\dagger(x)\frac{\lambda^a}{2}\psi(x) \ . \tag{16}$$

Thus we have arrived at a continuum formulation of non–abelian gauge theories entirely in terms of angular degrees of freedom. The objective for doing so was the wish to have a formulation that leaves us freedom in choosing appropriate unphysical "angle" variables. Since the introduction of appropriate coordinates is already very important in quantum mechanics, this freedom may be crucial for developing useful approximations

to understand the low energy properties of QCD. The formulation we found can be shown to be equivalent in a finite volume to Bars corner variable formulation. Therefore also the obvious similarity, in terms of variables, to the lattice hamiltonian approach to QCD [4] can be made precise by appropriately discretizing the spatial variables in our results [5]. Thus the "angle" variable formulation of QCD in the continuum shows a close relationship with the lattice QCD approach and it has the built–in freedom in selecting unphysical variables. Moreover for the axial gauge representation of QCD [6] it has been shown already that this reformulation leads to great technical simplifications in the elimination of unphysical variables [7]. All these advantages together may render the "angle" formulation an useful starting point for investigating non–perturbative aspects of QCD in terms of gauge invariant degrees of freedom.

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Appendix

From the definition of the angles $\xi_i(x)$ eq.(7) it is clear that the ξ_i only depend on the gauge fields A_i. Infinitesimal changes of the matrix V_i are related to changes in the "angles" ξ_i through

$$\begin{aligned} V_i^\dagger(x)\delta V_i(x) &= iM_i^{ab}(x)\frac{\lambda^b}{2}\delta\xi_i^a(x) && (17)\\ \delta\xi_i^b(x) &= -iW_i^{cb}(x)\mathrm{Tr}\left\{V_i^\dagger(x)\delta V_i(x)\lambda^c\right\} && (18)\\ \frac{\delta\xi_i^b(z)}{\delta A_j^a(x)} &= \mathrm{Tr}\left\{V_i^\dagger(z)\frac{-i\delta}{\delta A_j^a(x)}V_i(z)\lambda^c\right\}W_i^{cb}(z) && (19)\end{aligned}$$

with $W_i(x) = M_i(x)^{-1}$. The angular momentum operators are then defined by

$$J_k^c(z) = W_k^{cb}(z)\frac{-i\delta}{\delta\xi_k^b(z)} \tag{20}$$

and have commutation relations given in eqs.(10).

To obtain the representation eq.(12) for the electric field operator in terms of these "angular" momentum operators we have to make use of the following identity for the derivative of the path ordered integral V_i with respect to A_j

$$\frac{-i\delta V_i(z)}{\delta A_j^a(x)} = g\delta_{i,j}\delta^{d-1}(z_i^\perp - x_i^\perp)\theta(z_i - x_i)\theta(x_i)V_i(z)V_i^\dagger(x)\frac{\lambda^a}{2}V_i(x) \tag{21}$$

which may be verified by taking the derivative with respect to z_i [6].

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